Proposition 7.4:

Every monotonic function $f: [a,b] \longrightarrow \mathbb{R}$ is integrable.

Proof:

Let f be monotonically increasing (for monotonically decreasing f there is an analogous proof). Set

$$x_k := a + k \cdot \frac{b-a}{n}, \quad (k=0,1,\dots,n)$$

$\Rightarrow$ this gives an equidistant sub-division of $[a,b]$. With respect to this subdivision define step functions $\varphi, \psi \in S[a,b]$:

$$\varphi(x) := f(x_{k-1}) \text{ for } x_{k-1} \leq x < x_k,$$

$$\psi(x) := f(x_k) \quad \text{for } x_{k-1} \leq x < x_k,$$

and $\varphi(b) = \psi(b) = f(b)$. As f is monotonically increasing, we have

$$\varphi \leq f \leq \psi$$

and $\int_a^b \psi(x)\,dx - \int_a^b \varphi(x)\,dx$

$$= \sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}) - \sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1})$$

$$= \frac{(b-a)}{n}\left(\sum_{k=1}^{n} f(x_k) - \sum_{k=1}^{n} f(x_{k-1})\right)$$

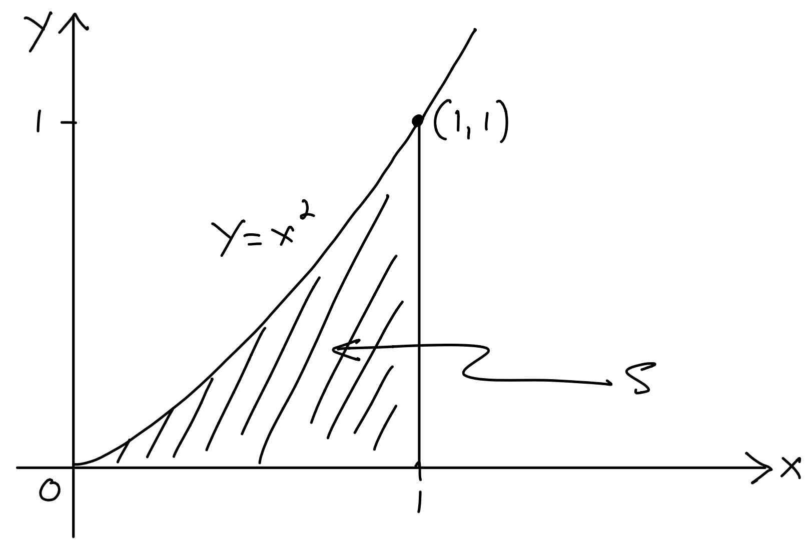
$$= \frac{b-a}{n}\left(f(x_n) - f(x_0)\right) \leq \varepsilon$$

for n big enough.

Prop. 7.2 $\Rightarrow$ f is integrable. $\square$

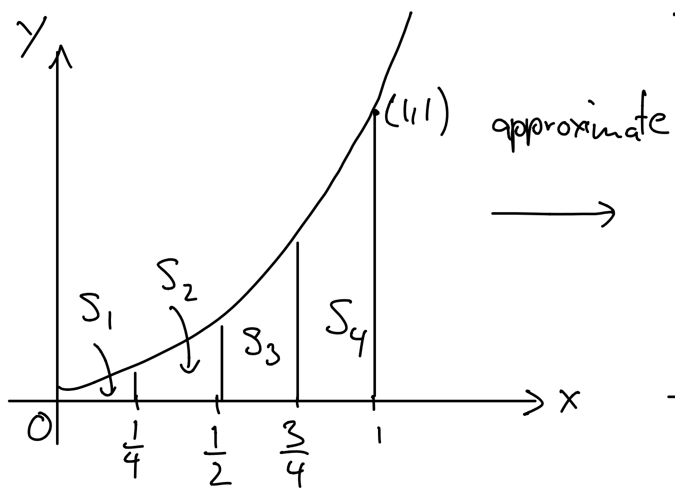
<u>Example 7.2:</u>

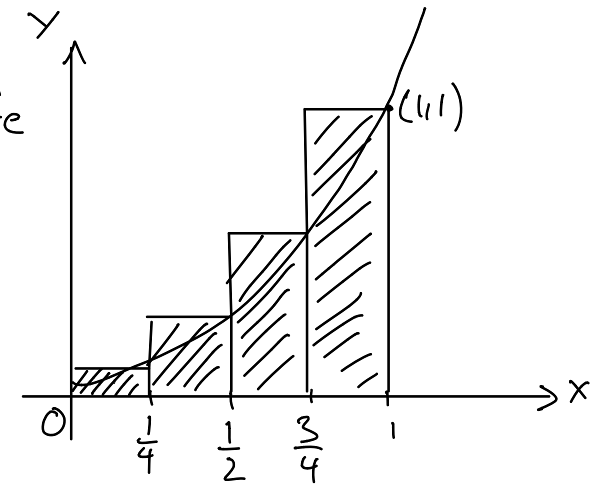
i) We use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1:



$\rightarrow$ We notice that the area of S must be between 0 and 1 (S is contained in square with side length 1).

$\rightarrow$ Improve by dividing S into 4 strips S_1, S_2, S_3, S_4:





For approximated area we then get:

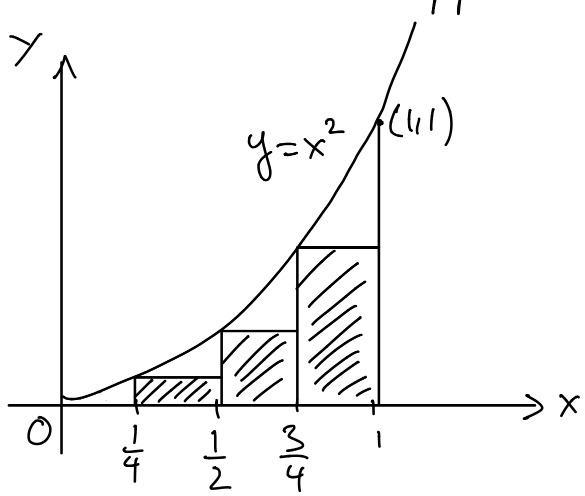
$$R_4 = \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 + \frac{1}{4}\cdot 1^2$$

$$= \frac{15}{32} = 0.46875$$

We also see from the figure that the actual area A of S is less than R_4:

$$A < 0.46875$$

$\longrightarrow$ Use second approximation:



The area is given here by:

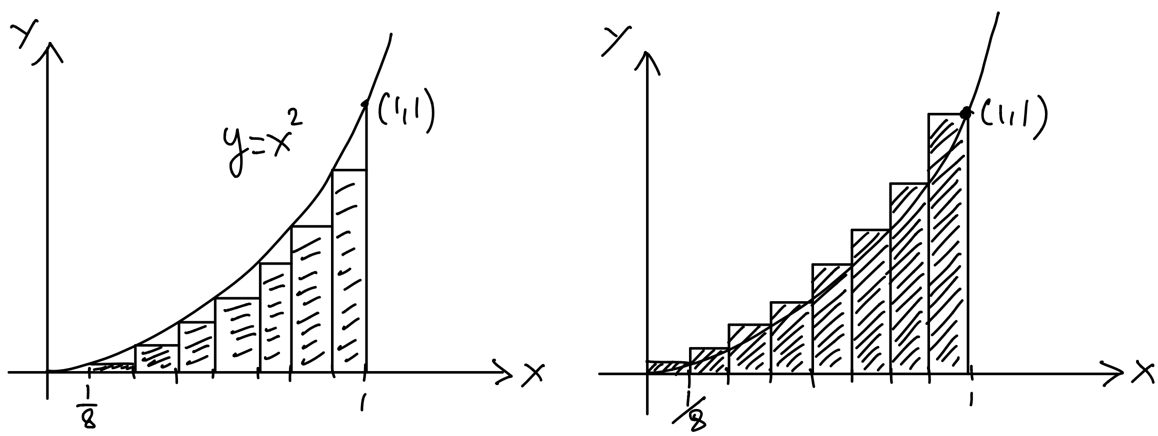
$$L_4 = \frac{1}{4}\cdot 0^2 + \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2$$

$$= \frac{7}{32} = 0.21875$$

→ get the following estimate for area of S:

$$0.21875 < A < 0.46875$$

→ repeat the procedure for larger number of strips:



Computing the sum of areas we now get the following estimate:

$$\underset{\substack{\| \\ 0.2734}}{L_8} < A < \underset{\substack{\| \\ 0.3984}}{R_8}$$

With 1000 strips we narrow it down to:

$$0.3328335 < A < 0.3338335$$

→ $A \approx 0.3333335$ is a good estimate

ii) We show that $\lim_{n \to \infty} R_n = \frac{1}{3}$

<u>Proof:</u>

We have $R_n = \sum_{i=1}^{n} r_i$, where $r_i = \frac{1}{n} \left(\frac{i}{n}\right)^2$

$$\Rightarrow R_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$

$$= \frac{1}{n} \cdot \frac{1}{n^2}\left(1^2 + 2^2 + 3^2 + \cdots + n^2\right)$$

$$= \frac{1}{n^3}\left(1^2 + 2^2 + 3^2 + \cdots + n^2\right)$$

Now we use

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \qquad (*)$$

(Can be shown by Induction)

Using $(*)$ we get

$$R_n = \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

Thus we have

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2}$$

$$= \lim_{n\to\infty} \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

$$= \frac{1}{6} \cdot 1 \cdot 2 = \frac{1}{3}$$

$\square$

From Prop. 7.4 we know that $\lim_{n\to\infty} L_n = \frac{1}{3}$ as well. Thus altogether we get

$$\int_0^1 x^2 \, dx = \frac{1}{3} = A$$

Definition 7.4 (Riemann sums):

Let $f: [a,b] \longrightarrow \mathbb{R}$ be a function,

$$a = x_0 < x_1 < \cdots < x_n = b$$

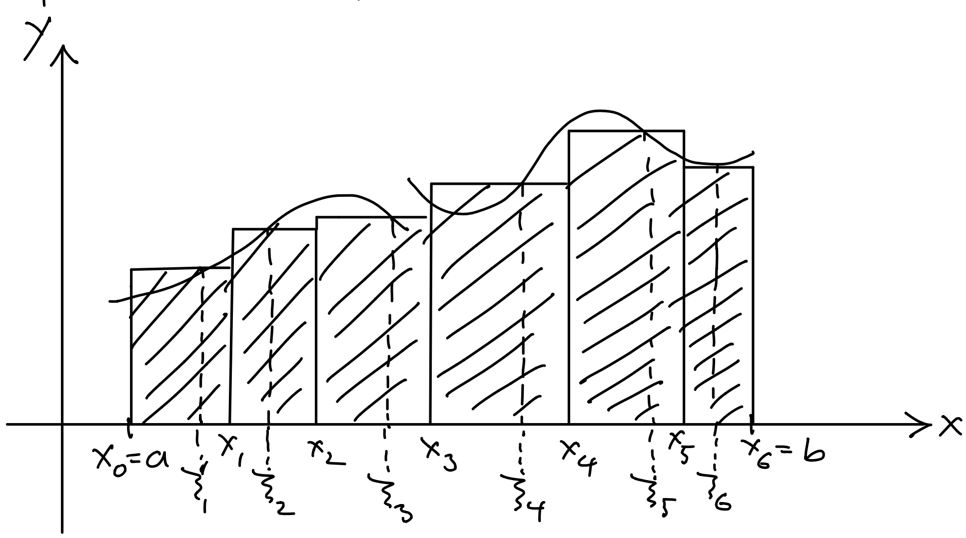
a sub-division of $[a,b]$ and ξ_k an arbitrary point on the interval $[x_{k-1}, x_k]$. We denote by

$$Z := \left((x_k)_{0 \le k \le n}, (\xi_k)_{1 \le k \le n} \right)$$

the set of x_k and ξ_k. Then

$$S(Z, f) := \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1})$$

is called the Riemann sum of the function f with respect to to Z. The Riemann sum is nothing else than the integral of the step function which approximates f at the points ξ_k.



The "fineness" of Z is defined as

$$\mu(Z) := \max_{1 \leq k \leq n} (x_k - x_{k-1})$$

<u>Theorem 7.1:</u>

Let $f: [a,b] \longrightarrow \mathbb{R}$ be a Riemann-integrable function. There there exists for every $\varepsilon > 0$ a $\delta > 0$, such that for every choice Z of points x_k and ξ_k of fineness $\mu(Z) < \delta$ we have:

$$\left| \int_a^b f(x)\,dx - S(Z,f) \right| \leq \varepsilon .$$

One can also write this as follows:

$$\lim_{\mu(Z) \to 0} S(Z,f) = \int_a^b f(x)\,dx.$$

For the proof we have to do some work.

<u>Proposition 7.5</u> (Linearity and Monotony):
Let $f, g: [a,b] \longrightarrow \mathbb{R}$ be integrable and $\lambda \in \mathbb{R}$. Then the functions $f+g$ and λf are also integrable and we have:

i) $\displaystyle\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

ii) $\int_a^b (\lambda f)(x)\,dx = \lambda \int_a^b f(x)\,dx.$

iii) $f \le g \implies \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$

Proof:

We want to use the criterion of Prop. 7.2

i) Let $\varepsilon > 0$ be given. Then we have step functions $\varphi_1, \psi_1, \varphi_2, \psi_2 \in S[a,b]$ s.t.

$$\varphi_1 \le f \le \psi_1, \quad \varphi_2 \le g \le \psi_2$$

and

$$\int_a^b \psi_1(x)\,dx - \int_a^b \varphi_1(x)\,dx \le \frac{\varepsilon}{2} \quad \text{and}$$

$$\int_a^b \psi_2(x)\,dx - \int_a^b \varphi_2(x)\,dx \le \frac{\varepsilon}{2}$$

Addition gives

$$\varphi_1 + \varphi_2 \le f + g \le \psi_1 + \psi_2$$

and

$$\int_a^b (\psi_1(x) + \psi_2(x))\,dx - \int_a^b (\varphi_1(x) + \varphi_2(x))\,dx \le \varepsilon.$$

$\implies f+g$ is integrable with integral given by i)

ii) The claim is trivial for $\lambda = 0$ and $\lambda = -1$

$\implies$ need to show for $\lambda > 0$. Let $\varepsilon > 0$ be given.

$\implies \exists$ step functions φ, ψ with $\varphi \le f \le \psi$ and

$$\int_a^b \psi(x)\,dx - \int_a^b \varphi(x)\,dx \le \frac{\varepsilon}{\lambda}$$

$\Longrightarrow$ $\lambda\varphi \le \lambda f \le \lambda\psi$ and

$$\int_a^b (\lambda\psi)(x)\,dx - \int_a^b (\lambda\varphi)(x)\,dx \le \varepsilon$$

From this ii) follows.

iii) trivial. $\square$